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Uniform Approximation and a Generalized Minimax Theorem

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1. INTRODUCTION

We characterize best uniform approximations by finite-dimensional subspaces of continuous functions from a compact Hausdorff space to a normed linear space. In the characterization we reveal usefulness of a minimax theorem presented in this paper.

We first prove this minimax theorem in general setup, deduce several corollaries from it, and show that it includes a classical minimax theorem given in [3]. Then we use the corollaries to derive necessary and sufficient conditions for best uniform approximations. Finally, we state the Haar condition in our framework, discuss the uniqueness of best approximations, and then relate our results to those obtained in Chebyshev approximation by real or complex polynomials [1, 2, 5, 6].

Such a derivation seems new. We hope that our minimax theorem can be applied to other areas of approximation theory.

2. BEST UNIFORM APPROXIMATION

Let X be a compact Hausdorff space and Y a normed linear space with norm $\|\cdot\|$. Let $C(X, Y)$ denote the set of all continuous functions from X to Y. Let A be an n-dimensional subspace of $C(X, Y)$ and $F \in C(X, Y)$. An element $f^* \in A$ is called a best (uniform) approximation to F if f^* minimizes over A

$$\max_{x \in X} \|f(x) - F(x)\|,$$

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that is, if

$$\max_{x \in X} \| f^*(x) - F(x) \| \leqslant \max_{x \in X} \| f(x) - F(x) \| \tag{2.1}$$

holds for all $f \in A$.

We are concerned with necessary and sufficient conditions and uniqueness of best approximations. These reflect various properties of best uniform (Chebyshev) approximation by real or complex polynomials [1, Chap. 3; 2, Chap. 7].

As is indicated by (2.1), f^* attains the minimax value of $\| f(x) - F(x) \|$. Therefore, this f^* and its corresponding counterpart constitute a saddle point (see (3.1)). From this consideration we can expect to characterize f^* via a minimax theorem. In view of the first part of Section 3, however, classical minimax theorems require the convexity (in f) and the concavity (in x) of $\| f(x) - F(x) \|$ and other conditions. These are too stringent to apply to the above problem. Hence, we need a minimax theorem which is applicable to approximation theory. This is the subject of the next section.

3. The Minimax Theorem

Let U and V be nonempty compact convex subsets of two Hausdorff topological vector spaces. Suppose that a function $J: U \times V \to R$ is such that for each $v \in V$, $J(\cdot, v)$ is lower semi-continous and convex on U, and for each $u \in U$, $J(u, \cdot)$ is upper semi-continuous and concave on V. Then, as is well known [3], there exists a saddle point $(u^*, v^*) \in U \times V$ such that

$$J(u^*, v) \leqslant J(u^*, v^*) \leqslant J(u, v^*), \qquad u \in U, v \in V, \tag{3.1}$$

that is,

$$\min_{u \in U} \max_{v \in V} J(u, v) = \max_{v \in V} \min_{u \in U} J(u, v).$$

However, if the set V is not convex, or if for some $u \in U$, $J(u, \cdot)$ is not a concave function on V, the relation (3.1) does not hold in general.

We present here a generalized minimax theorem that holds even under these conditions.

Let U be a nonempty compact convex subset of a Hausdorff topological vector space, and let V be an arbitrary nonempty set. Suppose that $J: U \times V \to R$ is such that for each $v \in V$, $J(\cdot, v)$ is a lower semi-continuous and convex function on U. For each positive integer n, define the set

$$\bar{V}_n = \Big\{ (\bar{\lambda}_n, \bar{v}_n) \mid \bar{\lambda}_n = (\lambda_1, \ldots, \lambda_n),\ \bar{v}_n = (v_1, \ldots, v_n),\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geqslant 0,$$
$$v_i \in V (i = 1, \ldots, n) \Big\}.$$

THEOREM 3.1. *Under the above assumptions*,

$$\min_{u \in U} \sup_{v \in V} J(u, v) = \lim_{n \to \infty} \sup_{(\bar{\lambda}_n, \bar{v}_n) \in \bar{V}_n} \min_{u \in U} \sum_{i=1}^{n} \lambda_i J(u, v_i).$$

Proof. Put

$$c = \lim_{n \to \infty} \sup_{(\bar{\lambda}_n, \bar{v}_n) \in \bar{V}_n} \min_{u \in U} \sum_{i=1}^{n} \lambda_i J(u, v_i)$$

and, for each n,

$$c_n = \sup_{(\bar{\lambda}_n, \bar{v}_n) \in \bar{V}_n} \min_{u \in U} \sum_{i=1}^{n} \lambda_i J(u, v_i).$$

Obviously, the sequence $\{c_n\}_{n=1}^{\infty}$ is monotone nondecreasing and $c = \lim_{n \to \infty} c_n$. For each $v \in V$, define the set

$$S_v = \{u \in U \mid J(u, v) \leqslant c\}.$$

Since $J(\cdot, v)$ is lower semi-continuous and convex, it is easy to see that S_v is a compact convex set. If we can prove that for any finite set $\{v_1, ..., v_k\}$ of V,

$$\bigcap_{i=1}^{k} S_{v_i} \neq \varnothing, \tag{3.2}$$

then, using the fact that U is a compact set, we conclude that

$$\bigcap_{v \in V} S_v \neq \varnothing.$$

Then there is $u^* \in \bigcap_{v \in V} S_v$, which means that $J(u^*, v) \leqslant c$ for all $v \in V$. Hence

$$\sup_{v \in V} J(u^*, v) \leqslant c. \tag{3.3}$$

On the other hand, it follows from the well-known relation $\inf \sup J \geqslant \sup \inf J$ that

$$c_n \leqslant \inf_{u \in U} \sup_{(\bar{\lambda}_n, \bar{v}_n) \in \bar{V}_n} \sum_{i=1}^{n} \lambda_i J(u, v_i) = \inf_{u \in U} \sup_{v \in V} J(u, v).$$

Therefore, we have

$$c \leqslant \inf_{u \in U} \sup_{v \in V} J(u, v)$$

by $c = \lim_{n \to \infty} c_n$. From this and (3.3), we get

$$\sup_{v \in V} J(u^*, v) = \min_{u \in U} \sup_{v \in V} J(u, v) = c.$$

This is the desired result. Therefore, it suffices to prove (3.2). Consider the system of convex inequalities

$$J(u, v_j) \leqslant c, \qquad j = 1, \ldots, k. \tag{3.4}$$

We show that there exists a u satisfying (3.4). Let $(\mu_1, \ldots, \mu_k)$ be an arbitrary set of nonnegative real numbers with $\sum_{j=1}^{k} \mu_j = 1$. Define the function f by

$$f(u) = \sum_{j=1}^{k} \mu_j J(u, v_j).$$

By the definition of c_k, it follows that there exists a $\bar{u} \in U$ satisfying $f(\bar{u}) \leqslant c_k$, hence $f(\bar{u}) \leqslant c$. Since $(\mu_1, \ldots, \mu_k)$ is arbitrary, we conclude that the system (3.4) is consistent on U (see [4, Theorem 1]), i.e., there exists $\bar{u} \in U$ such that $J(\bar{u}, v_j) \leqslant c$, $j = 1, \ldots, k$. This is equivalent to the relation (3.2). Hence the proof is complete.

In the case V is a convex subset of another Hausdorff topological vector space, we get the well-known minimax theorem mentioned above.

COROLLARY 3.1. *Suppose that the assumptions of Theorem* 3.1 *are fulfilled. Furthermore, assume that V is a convex subset of another Hausdorff topological vector space and $J(u, \cdot)$ is a concave function of v for each $u \in U$. Then*

$$\min_{u \in U} \sup_{v \in V} J(u, v) = \sup_{v \in V} \min_{u \in U} J(u, v).$$

Proof. For each n we have

$$\sup_{(\lambda_n, v_n) \in \Gamma_n} \min_{u \in U} \sum_{i=1}^{n} \lambda_i J(u, v_i) \leqslant \sup_{(\lambda_n, v_n) \in \Gamma_n} \min_{u \in U} J\left(u, \sum_{i=1}^{n} \lambda_i v_i\right)$$

since $J(u, \cdot)$ is concave, and the right-hand side is equal to

$$\sup_{v \in V} \min_{u \in U} J(u, v),$$

by convexity of V. Theorem 3.1 implies

$$\min_{u \in U} \sup_{v \in V} J(u, v) \leqslant \sup_{v \in V} \min_{u \in U} J(u, v).$$

Since the reverse inequality always holds, the corollary follows.

If U is finite dimensional, Theorem 3.1 takes on the following simple form.

COROLLARY 3.2. *In Theorem 3.1, if U is an n-dimensional compact convex subset of a Hausdorff topological vector space,*

$$\min_{u \in U} \sup_{v \in V} J(u, v) = \sup_{(\bar{\lambda}, \bar{v}) \in \bar{V}_{n+1}} \min_{u \in U} \sum_{i=1}^{n+1} \lambda_i J(u, v_i).$$

Proof. Let c be the value in the right-hand side and define the sets S_v as in the proof of Theorem 3.1. In order to prove $\bigcap_{v \in V} S_v \neq \varnothing$, we can apply Helly's theorem [1], since U is finite dimensional. That is, it suffices only to prove

$$\bigcap_{i=1}^{n+1} S_{v_i} \neq \varnothing$$

for any $n+1$ elements $v_1, ..., v_{n+1}$ of V. However, this part of the proof is the same as that of Theorem 3.1, hence we omit it.

In the next corollary we assume that the set V has a topology for which the function J is jointly continuous.

COROLLARY 3.3. *Let U be an n-dimensional, compact convex subset of a Hausdorff topological vector space ($n \geqslant 1$), V a compact Hausdorff space. Let $J: U \times V \to R$ be a jointly continuous function. Then, $u^* \in U$ minimizes* $\max_{v \in V} J(u, v)$ *over U if and only if there exists $(\bar{\lambda}^*_{n+1}, \bar{v}^*_{n+1}) \in \bar{V}_{n+1}$* such that

$$\sum_{i=1}^{n+1} \lambda_i J(u^*, v_i) \leqslant \sum_{i=1}^{n+1} \lambda_i^* J(u^*, v_i^*) \leqslant \sum_{i=1}^{n+1} \lambda_i^* J(u, v_i^*) \tag{3.5}$$

holds for all $(\bar{\lambda}_{n+1}, \bar{v}_{n+1}) \in \bar{V}_{n+1}$ and for all $u \in U$.

Proof. First note that the function

$$\min_{u \in U} \sum_{i=1}^{n+1} \lambda_i J(u, v_i)$$

is continuous with respect to $\lambda_1, ..., \lambda_{n+1}; v_1, ..., v_{n+1}$, and that the set $\bar{V}_{n+1}$ is compact. Hence there exists $(\bar{\lambda}^*_{n+1}, \bar{v}^*_{n+1}) = (\lambda^*_1, ..., \lambda^*_{n+1}, v^*_1, ..., v^*_{n+1}) \in \bar{V}_{n+1}$ such that

$$\max_{(\bar{\lambda}_{n+1}, \bar{v}_{n+1}) \in \bar{V}_{n+1}} \min_{u \in U} \sum_{i=1}^{n+1} \lambda_i J(u, v_i) = \min_{u \in U} \sum_{i=1}^{n+1} \lambda_i^* J(u, v_i^*).$$

By hypothesis, u^* satisfies

$$\min_{u \in U} \max_{v \in V} J(u, v) = \max_{v \in V} J(u^*, v) = \max_{(\lambda_{n+1}, v_{n+1}) \in V_{n+1}} \sum_{i=1}^{n+1} \lambda_i J(u^*, v_i).$$

These two relations and the preceding corollary imply (3.5). Conversely, (3.5) together with Corollary 3.2 implies

$$\begin{aligned} &\min_{u \in U} \max_{v \in V} J(u, v) \\ &\quad = \max_{(\lambda_{n+1}, v_{n+1}) \in V_{n+1}} \min_{u \in U} \sum_{i=1}^{n+1} \lambda_i J(u, v_i) \\ &\quad = \max_{(\lambda_{n+1}, v_{n+1}) \in V_{n+1}} \sum_{i=1}^{n+1} \lambda_i J(u^*, v_i) = \max_{u \in U} J(u^*, v). \end{aligned}$$

Therefore u^* minimizes $\max_{v \in V} J(u, v)$. This completes the proof.

4. Necessary and Sufficient Conditions

We continue to adopt the notation employed in Section 2. For $f \in C(X, Y)$, we define the uniform norm of f by

$$|||f||| = \max_{x \in X} \|f(x)\|,$$

and endow the linear space $C(X, Y)$ with the uniform topology.

It is easy to see that $\|f(x) - F(x)\|$ is a jointly continuous function of the two variables f, x and convex in f, i.e.,

$$\|(\theta f + (1-\theta) g)(x) - F(x)\| \leqslant \theta \|f(x) - F(x)\| + (1-\theta) \|g(x) - F(x)\|$$

for all f, $g \in C(X, Y)$, $x \in X$, and θ, $0 \leqslant \theta \leqslant 1$. Hence, if $A \subset C(X, Y)$ is a finite-dimensional subspace, we can apply Corollary 3.3 and obtain the following necessary and sufficient condition of best approximation.

Theorem 4.1. *Let A be an n-dimensional subspace of $C(X, Y)$ and $F \in C(X, Y)$. Then $f^* \in A$ is a best uniform approximation to F if and only if there exist $\lambda_1^*, \ldots, \lambda_k^* > 0$, $\sum_{i=1}^k \lambda_i^* = 1$, and k distinct elements $x_1^*, \ldots, x_k^*$ of X, where $1 \leqslant k \leqslant n+1$, satisfying*

(i) $\|f^*(x_i^*) - F(x_i^*)\| = |||f^* - F|||$, $\quad i = 1, \ldots, k$;

(ii) $\sum_{i=1}^k \lambda_i^* \|f(x_i^*) - F(x_i^*)\| \geqslant \sum_{i=1}^k \lambda_i^* \|f^*(x_i^*) - F(x_i^*)\|$ for all $f \in A$.

Proof. Let f^* be a best approximation and let $U = \{f \in A \mid |||f^* - f||| \leqslant 1\}$. It is evident that f^* minimizes $\max_{x \in X} \|f(x) - F(x)\|$ over U, and U is a compact subset of A because it is bounded and closed in a finite-dimensional space A. Applying Corollary 3.3 yields the existence of $\lambda'_1, ..., \lambda'_{n+1} \geqslant 0$, $\sum_{i=1}^{n+1} \lambda'_i = 1$, and $\{x'_1, ..., x'_{n+1}\} \subset X$ such that the following two relations hold:

$$\sum_{i=1}^{n+1} \lambda'_i \|f^*(x'_i) - F(x'_i)\| \geqslant \sum_{i=1}^{n+1} \lambda_i \|f^*(x_i) - F(x_i)\| \tag{4.1}$$

for all $(\bar{\lambda}_{n+1}, \bar{x}_{n+1}) \in \bar{X}_{n+1} = \{(\bar{\lambda}_{n+1}, \bar{x}_{n+1}) \mid \bar{\lambda}_{n+1} = (\lambda_1, ..., \lambda_{n+1}),\ \bar{x}_{n+1} = (x_1, ..., x_{n+1}),\ \sum_{i=1}^{n+1} \lambda_i = 1,\ \lambda_i \geqslant 0,\ x_i \in X\ (i = 1, ..., n+1)\}$;

$$\sum_{i=1}^{n+1} \lambda'_i \|f(x'_i) - F(x'_i)\| \geqslant \sum_{i=1}^{n+1} \lambda'_i \|f^*(x'_i) - F(x'_i)\| \tag{4.2}$$

for all $f \in U$. Let us denote by $\lambda_1^*, ..., \lambda_k^*$ the nonzero elements within $\lambda'_1, ..., \lambda'_{n+1}$ and by $x_1^*, ..., x_k^*$ the corresponding elements within $x'_1, ..., x'_{n+1}$ of X. The assertion (i) follows from (4.1) which means, for $i = 1, ..., k$,

$$\|f^*(x_i^*) - F(x_i^*)\| = \max_{x \in X} \|f^*(x) - F(x)\| = |||f^* - F|||.$$

On the other hand, it follows from (4.2) that

$$\sum_{i=1}^{k} \lambda_i^* \|f(x_i^*) - F(x_i^*)\| \geqslant \sum_{i=1}^{k} \lambda_i^* \|f^*(x_i^*) - F(x_i^*)\|$$

holds for all $f \in U$. Since the left-hand side is a convex function of f and has a local minimum at f^*, f^* realizes a global minimum by a property of convex functions. Thus (ii) follows. Conversely, suppose that (i) and (ii) hold. These two conditions yield

$$\sup_{(\bar{\lambda}_k, \bar{x}_k) \in \bar{X}_k} \inf_{f \in A} \sum_{i=1}^{k} \lambda_i \|f(x_i) - F(x_i)\|$$
$$\geqslant \sum_{i=1}^{k} \lambda_i^* \|f^*(x_i^*) - F(x_i^*)\| = |||f^* - F|||,$$

where $\bar{X}_k$ is similarly defined as the set $\bar{X}_{n+1}$. The left-hand side of the last relation is equal to or less than

$$\inf_{f \in A} \max_{(\bar{\lambda}_k, \bar{x}_k) \in \bar{X}_k} \sum_{i=1}^{k} \lambda_i \|f(x_i) - F(x_i)\| = \inf_{f \in A} \max_{x \in X} \|f(x) - F(x)\|.$$

Therefore f^* is a best approximation. The proof is completed.

If Y is a Hilbert space with inner product $\langle , \rangle$, then condition (ii) of Theorem 4.1 can be replaced by another form.

COROLLARY 4.1. *$f^* \in A$ is a best uniform approximation to F if and only if there exist $\lambda_1^*,..., \lambda_k^* > 0$, $\sum_{i=1}^k \lambda_i^* = 1$, and k distinct elements $x_1^*,..., x_k^* \in X$, where $1 \leqslant k \leqslant n+1$, satisfying condition* (i) *of Theorem* 4.1 *and*

$$\text{(iii)} \qquad \sum_{i=1}^{k} \lambda_i^* \langle f^*(x_i^*) - F(x_i^*), p(x_i^*) \rangle = 0 \qquad \textit{for all } p \in A.$$

Proof. Condition (ii) of Theorem 4.1 holds if and only if the following inequality holds for all $p \in A$ and all real numbers t:

$$\sum_{i=1}^{k} \lambda_i^* \| f^*(x_i^*) + tp(x_i^*) - F(x_i^*)\| \geqslant \sum_{i=1}^{k} \lambda_i^* \| f^*(x_i^*) - F(x_i^*)\|. \quad (4.3)$$

This means that the left-hand side is a convex function of t and has a global minimum at $t = 0$. By differentiating it with respect to t at $t = 0$, it follows that (iii) is a necessary and sufficient condition for (4.3) to hold for all $p \in A$ and t. For we have, for $y, z \in Y$ and real t,

$$\lim_{t \to 0} \frac{\| y + tz \| - \| y \|}{t} = \lim_{t \to 0} \frac{\| y + tz \|^2 - \| y \|^2}{t(\| y + tz \| + \| y \|)}$$

$$= \frac{\langle y, z \rangle}{\| y \|}, \qquad \text{if } y \neq 0.$$

This completes the proof.

The next corollary states that f^* is a best approximation on X if and only if it also is on some finite set of X.

COROLLARY 4.2. *Let A be an n-dimensional subspace of $C(X, Y)$ and $F \in C(X, Y)$. Then f^* is a best uniform approximation to F if and only if there exist l elements $z_1^*,..., z_l^* \in X$, where $1 \leqslant l \leqslant n+1$, for which f^* satisfies*

$$\text{(i}'\text{)} \quad \| f^*(z_j^*) - F(z_j^*)\| = \| f^* - F \|, \qquad j = 1,..., l;$$

$$\text{(ii}'\text{)} \quad \max_{1 \leqslant j \leqslant l} \| f(z_j^*) - F(z_j^*)\| \geqslant \max_{1 \leqslant j \leqslant l} \| f^*(z_j^*) - F(z_j^*)\| \qquad \text{for all } f \in A.$$

Proof. The proof is an easy application of Theorem 4.1, hence we omit it.

As an example of applications of the above results, we take $C(X, Y)$ to be complex-valued continuous functions on a compact subset of the complex plane. Let us take the subspace A to be the set of all complex polynomials of degrees at most n. It is easy to see that Corollary 4.1 implies

Characterization Theorem [1, Chap. 3] and that Corollary 4.2 corresponds to Theorem C (Skeleton Theorem) of [6].

5. THE HAAR CONDITION

In this section we introduce a condition under which the numbers k, l appearing in Theorem 4.1 and its corollaries are equal to exactly $n+1$. The n-dimensional subspace $A \subset C(X, Y)$ is said to be a Haar subspace if for any n distinct elements $\{x_1, \ldots, x_n\} \subset X$ and for any $\{y_1, \ldots, y_n\} \subset Y$, there exists a unique $f \in A$ such that $f(x_i) = y_i$, $i = 1, \ldots, n$. When $C(X, Y)$ is $C[a, b]$, this is equivalent to saying that A satisfies the Haar condition [5, p. 91].

THEOREM 5.1. *Let X be a compact Hausdorff space that contains more than n points and A an n-dimensional Haar subspace of $C(X, Y)$. Let $F \in C(X, Y)$ with $F \notin A$. $f^* \in A$ is a best uniform approximation to F if and only if there exist $\lambda_1^*, \ldots, \lambda_{n+1}^* > 0$, $\sum_{i=1}^{n+1} \lambda_i^* = 1$, and $n+1$ distinct elements $x_1^*, \ldots, x_{n+1}^* \in X$ satisfying*

$$\text{(i)} \quad \| f^*(x_i^*) - F(x_i^*) \| = \| f^* - F \|, \qquad i = 1, \ldots, n+1;$$

$$\text{(ii)} \quad \sum_{i=1}^{n+1} \lambda_i^* \| f(x_i^*) - F(x_i^*) \| \geq \sum_{i=1}^{n+1} \lambda_i^* \| f^*(x_i^*) - F(x_i^*) \|$$

for all $f \in A$,

or if and only if there exist $n+1$ distinct elements $x_1^*, \ldots, x_{n+1}^* \in X$ satisfying (i) *and*

$$\text{(ii}') \quad \max_{1 \leq i \leq n+1} \| f(x_i^*) - F(x_i^*) \| \geq \max_{1 \leq i \leq n+1} \| f^*(x_i^*) - F(x_i^*) \|$$

for all $f \in A$. Furthermore, if the normed linear space Y is strictly convex, there exists a unique best approximation.

Proof. If the numbers k or l in Theorem 4.1 or Corollary 4.2 are less than $n+1$, we can choose an $\bar{f} \in A$ such that

$$\bar{f}(x_i^*) = F(x_i^*) (i = 1, \ldots, k, k \leq n), \qquad \text{or} \qquad \bar{f}(z_j^*) = F(z_j^*) (j = 1, \ldots, l, l \leq n).$$

It follows from (ii) of Theorem 4.1 that

$$0 = \sum_{i=1}^{k} \lambda_i^* \| \bar{f}(x_i^*) - F(x_i^*) \| \geq \sum_{i=1}^{k} \lambda_i^* \| f^*(x_i^*) - F(x_i^*) \|$$

and from (ii′) of Corollary 4.2 that

$$0 = \max_{1 \leqslant j \leqslant l} \| \bar{f}(z_j^*) - F(z_j^*) \| \geqslant \max_{1 \leqslant j \leqslant l} \| f^*(z_j^*) - F(z_j^*) \| .$$

The right-hand sides of these inequalities are equal to $\| f^* - F \| > 0$, since $F \notin A$. This is a contradiction. Hence we have $k = l = n + 1$. Along the same argument of [2, p. 143], the uniqueness follows from the definitions of strict convexity [2, p. 141; or 5, p. 106] and Haar subspaces. This completes the proof.

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